Brief paper

Geometric characterization on the solvability of regulator equations☆

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Abstract

The solvability of the regulator equation for a general nonlinear system is discussed in this paper by using geometric method. The 'feedback' part of the regulator equation, that is, the feasible controllers for the regulator equation, is studied thoroughly. The concepts of minimal output zeroing control invariant submanifold and left invertibility are introduced to find all the possible controllers for the regulator equation under the condition of left invertibility. Useful results, such as a necessary condition for the output regulation problem and some properties of friend sets of controlled invariant manifolds, are also obtained.
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Keywords: Output regulation; Regulator equation; Controlled invariant submanifold; Friend set; Left invertibility

1. Introduction

The output regulation problem for the following nonlinear system

$$\dot{x} = F(x, w, u), \quad \dot{w} = s(w), \quad e = h(x, w), \tag{1}$$

has drawn much attention (see Byrnes, Priscoli, Isidori, & Kang, 1997; Chen & Huang, 2004, 2005a, 2005b; Cheng, Tarn, & Spurgeon, 2001; Huang, 2001, 2003; Huang & Chen, 2004; Huang & Lin, 1995; Huang & Rugh, 1992a, 1992b; Isidori, 1997; Knobloch, Isidori, & Flockerzi, 1993; Marconi & Isidori, 2000; Marconi, Isidori, & Serrani, 2004; Serrari & Isidori, 2000; Zheng, Zhang, & Evans, 2000) since the publication of the celebrated paper (Isidori & Bynes, 1990) by geometric method, where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^q$, $u(t) \in \mathbb{R}^m$, $e(t) \in \mathbb{R}^p$, $F(0, 0, 0) = 0$, $s(0) = 0$, $h(0, 0) = 0$. Under some necessary hypotheses, Isidori and Bynes (1990) and Isidori (1995) transfer the output regulation problem into the solvability of the following regulator equation:

$$\frac{\partial \pi}{\partial w} s(w) = F(\pi(w), w, u(w)), \quad 0 = h(\pi(w), w). \tag{2}$$

A solution to the above regulator equation consists of two parts, a 'state variable' part that defines a controlled invariant manifold, and a 'feedback' part that makes the manifold invariant. The state variable part can be stated as the algebraic equations that $\pi(w)$ must satisfy, while the feedback part is the set of all the feasible controllers $u(w)$'s which ensure the solvability of the regulator equation. The main attention in literature focuses on the state variable part in order to reduce the number of unknown states in (2), while the feedback part of (2) has not been well studied.

For the affine case $F = f(x) + g(x)u + p(x)w$, Isidori and Bynes (1990) gives a geometric characterization on the solvability of (2). Later, general cases are considered in Huang and Rugh (1992a), Cheng et al. (2001), Huang (2003), and Huang and Lin (1995) by analytic method. Some of the results require the existence of relative degrees. The present paper generalizes the geometric ideas in Isidori and Bynes (1990) to general nonlinear systems, discusses the feedback part of (2), and thus offers some new insight from differential geometric view point. By computing the defining equations of the maximal output zeroing submanifold, (2) is reduced into a center manifold equation with less number of unknowns (both state variables

and inputs). During the reduction, all the feasible controllers are found by some parametrization, and the process does not require the existence of relative degrees. Furthermore, an easily checkable necessary condition for the solvability of the output regulation problem is obtained in Proposition 1 too. After the work on the state variable part of (2), the second part of the paper discusses the feedback part of (2) by introducing a new concept of *left invertibility*. When a nonlinear system (1) is left invertible, all the feasible controllers are shown to be contained in the friend set of a minimal output zeroing control submanifold under some nonzero intersection condition.

The paper is organized as follows. Section 2 gives an algorithm to compute the maximal output zeroing submanifold for a general nonlinear system. Section 3 uses the algorithm to reduce the number of unknowns in the regulator equation (2). Section 4 discusses the parametrization of feasible controllers for the regulator equation by left invertibility. The last section is the conclusion.

For any set X, let X^c denote its connected component which contains the origin. All the functions in the paper are supposed to be smooth. The terminology *feasible controllers* of a solution manifold N of (2) refers to the friend set of N. The rank of a matrix, whose elements are functions, on an open set is defined to be the constant which equals the rank of the matrix at any point in the open set.

2. Computation of output zeroing submanifold

Suppose $f(0, 0) = 0$ and $h(0) = 0$ in the following system

$$\dot{x} = f(x, u), \quad y = h(x). \tag{3}$$

Definition 1. A connected submanifold $M = \{x : \phi(x) = 0\}$, which contains also the origin, is called controlled invariant on an open neighborhood U of 0 with respect to the system (3) if there exists a smooth function $\alpha(x)$ such that $\frac{\partial \phi}{\partial x} f(x, \alpha(x)) = 0$, or $f(x, \alpha(x)) \in T_x M$, for all $x \in M \cap U$. A controlled invariant submanifold, which is contained also in $\{x : h(x) = 0\}$, is called (locally) maximal if it is maximal with respect to the relation of inclusion. In this case, it is also called a (locally) maximal output zeroing submanifold. The corresponding $\alpha(x)$ is called a friend of M on the set U, and the set of all friends of M on U is denoted by $\mathscr{F}(M \cap U)$.

By Isidori (1995), the following algorithm computes the maximal output zeroing submanifold for (3).

Algorithm 1. *Input*: $M_0 = \{x : h(x) = 0\}$, $k = 1$.
  Output: k^* such that $M_{k^*}^c = M_{k^*+1}^c$.

(1) Let M_k be the set $\{x \in M_{k-1}^c : \text{there exists a smooth } u(x)$
  such that $f(x, u) \in T_x M_{k-1}^c\}$.
(2) If $M_k^c = M_{k-1}^c$ then let $k^* = k - 1$ and stop; otherwise repeat the above Step (1) for $k := k + 1$.

Remark 1. The function $u = u(x)$ for M_k may be different from that of M_{k-1}, and it is reasonable to denote the u for M_k

by u_k. Lemma 1 shows that once u_k is known, the function u for M_i, $i \leqslant k - 1$, can be chosen as u_k.

Lemma 1. *Let $u_k(x)$ be the function $u(x)$ defined in the definition of M_k in Algorithm 1, then $f(x_0, u_k(x_0)) \in T_{x_0} M_i$, $i = 0, \ldots, k$, where x_0 is any point in some open subset of M_k^c which contains the origin.*

Proof. It is obvious that $x_0 \in M_k^c \subseteq M_{k-1}^c \subseteq \cdots \subseteq M_0^c$. By $M_{i+1}^c \subseteq M_i^c$, $0 \leqslant i \leqslant k - 1$, one has $T M_{i+1}^c \subseteq T M_i^c$ locally. It is always possible to find a small enough open set U of M_k^c which contains the origin, and at the same time, for any $x_0 \in U$ there is $f(x_0, u_k(x_0)) \in T M_k^c \subseteq T M_{k-1}^c \subseteq \cdots \subseteq T M_0^c$. $\square$

Remark 2. Let $\mathscr{U}_k$ be the set of u which satisfies the condition of M_k in Algorithm 1, then Lemma 1 tells that $\mathscr{U}_k \subseteq \mathscr{U}_{k-1} \subseteq \cdots \subseteq \mathscr{U}_0$ holds in some neighborhood of the origin. This implies that, if $\mathscr{U}_k$ is the set of functions which satisfy some system of algebraic equations, then all the function of $\mathscr{U}_{k+1}$ must satisfy the same system of equations and, possibly, some other equations.

Now consider the details of Algorithm 1. Let $M_0 = \{x : H_0(x) := h(x) = 0\}$, then $M_1 = \{x \in M_0^c$, there exists a function $u = u(x)$ such that $\frac{\partial H_0}{\partial x} f(x, u) = 0\}$. Let $F_1(x, u) = \frac{\partial H_0}{\partial x} f(x, u)$ and suppose rank $\frac{\partial F_1}{\partial u}$ is a constant r_1 on the intersection of a small neighborhood of the origin and M_1. Rearranging the components of F_1 and u, one can assume $F_1 = ((F_1^1)^T, (F_1^2)^T)^T$, $u = (u_1^T, u_2^T)^T$, and $r_1 = \text{rank} \frac{\partial F_1^1}{\partial u_1}$. There exists a smooth matrix function $P(x, u)$ on a possibly smaller open neighborhood of the origin such that $\frac{\partial F_1^2}{\partial u_1} = P \frac{\partial F_1^1}{\partial u_1}$ and $\frac{\partial F_1^2}{\partial u_2} = P \frac{\partial F_1^1}{\partial u_2}$. By the Implicit Function Theorem, there exists a function α such that $u_1 = \alpha(u_2, x)$ ensures $F_1^1(x, \alpha(u_2, x), u_2) \equiv 0$ for all (x, u_2) in some neighborhood U' of 0, where $F_1^1(x, u_1, u_2)$ is just the function $F_1^1(x, u)$. Thus $F_1(x, u) = 0$ is solvable with respect to the unknown u if and only if there exists a function $u_2(x)$ which solves the equation $F_1^2(x, \alpha(u_2, x), u_2) = 0$.

Lemma 2. *Fix the notations above, and let $F_1^4(x, u_2) = F_1^2(x, \alpha(u_2, x), u_2)$, then $\frac{\partial F_1^4}{\partial u_2} \equiv 0$ for all $(x, u_2) \in U'$.*

Proof. Let $F_1^3(x, u_2) = F_1^1(x, \alpha(u_2, x), u_2)$, then $F_1^3 \equiv 0$ for all $(x, u_2) \in U'$. By the existence of the matrix P mentioned above one has $\frac{\partial F_1^4}{\partial u_2}(x, u_2) = \frac{\partial F_1^2}{\partial u_1} \frac{\partial \alpha}{\partial u_2} + \frac{\partial F_1^2}{\partial u_2} = P \frac{\partial F_1^1}{\partial u_1} \frac{\partial \alpha}{\partial u_2} + P \frac{\partial F_1^1}{\partial u_2} = P \frac{\partial F_1^3}{\partial u_2} \equiv 0$. $\square$

Lemma 2 shows that $F_1^4(x, u_2)$ does not contain u_2, and hence it can be written as $F_1^4(x)$ (see Example 1). Therefore there exists a function u such that $F_1(x, u) = 0$ is solvable for $x \in M_0^c$ if and only if $F_1^4(x) := F_2(x, \alpha(u_2, x), u_2) = 0$ is solvable on M_0^c. Then $M_1 = \{x \in M_0^c : F_1^4(x) = 0\}^c$. By letting $H_1(x) = ((H_0(x))^T, (F_1^4(x))^T)^T$, one has also $M_1 = \{x : H_1(x) = 0\}^c$. Let $\alpha_1(u_2, x) = \alpha(u_2, x)$, $\Gamma_1 = u_2$, $\Delta_1 = u_1$, then $\Delta_1 = \alpha_1(u_2, x) = \alpha_1(\Gamma_1, x)$. To compute M_2 it suffices to solve the equation

$\frac{\partial H_1(x)}{\partial x} f(x, u) = 0$. By Lemma 1 and Remark 2, the function u which ensures this equation must satisfy also $\Delta_1 = \alpha_1(\Gamma_1, x)$. Let $v = \Gamma_1$, $\widetilde{f}(x, v) = f(x, u)|_{\Delta_1 = \alpha_1(v, x)} = f(x, \alpha_1(v, x), v)$, $F_2(x, v) = \frac{\partial H_1(x)}{\partial x} \widetilde{f}(x, v)$, $r_2 = \text{rank} \frac{\partial F_2}{\partial v}$, and rearrange the components of F_2 and v, one has $F_2 = ((F_2^1)^{\mathrm{T}}, (F_2^2)^{\mathrm{T}})^{\mathrm{T}}$, $v = (v_1^{\mathrm{T}}, v_2^{\mathrm{T}})^{\mathrm{T}}$, $r_2 = \text{rank} \frac{\partial F_2^1}{\partial v_1}$. A similar computation gives some $F_2^4(x)$, $H_2 = (H_1^{\mathrm{T}}, (F_2^4)^{\mathrm{T}})^{\mathrm{T}}$, and $M_2 = \{x : H_2(x) = 0\}^c$. Thus the following algorithm is obtained.

Algorithm $1'$. *Input*: $M_0 = \{x : h(x) = 0\}^c$, $\Gamma_0 = (u_1, \ldots, u_m)^{\mathrm{T}}$, $\Delta_0 = \emptyset$, $\widetilde{f}_0(x, \Gamma_0) = f(x, u)$, $H_0(x) = h(x)$, $k = 1$. *Output*: k^*, H_{k^*}, Γ_{k^*}, $\Delta_{k^*} = \alpha_{k^*}(\Gamma_{k^*}, x)$.

(1) *Let* $v = \Gamma_{k-1}$, $F_k(x, v) = \frac{\partial H_{k-1}(x)}{\partial x} \widetilde{f}_{k-1}(x, v)$, *and* rank $\frac{\partial F_k(x, v)}{\partial v}$ *be* r_k *on a small open neighborhood of* 0.
(2) *Rearrange the components of* F_k *and* v *and decompose them into* $F_k = ((F_k^1)^{\mathrm{T}}, (F_k^2)^{\mathrm{T}})^{\mathrm{T}}$, $v = (v_1^{\mathrm{T}}, v_2^{\mathrm{T}})^{\mathrm{T}}$, *such that* rank $\frac{\partial F_k^1}{\partial v_1} = r_k$. *Find, by the Implicit Function Theorem, the function* $v_1 = \alpha(v_2, x)$ *such that* $F_k^1(x, v)|_{v_1 = \alpha(v_2, x)} = F_k^1(x, \alpha(v_2, x), v_2) \equiv 0$.
(3) *Let* $F_k^4(x) = F_k^2(x, \alpha(v_2, x), v_2)$. *If* rank $((\frac{\partial H_{k-1}(x)}{\partial x})^{\mathrm{T}},$ $(\frac{\partial F_k^4(x)}{\partial x})^{\mathrm{T}}) = \text{rank} \frac{\partial H_{k-1}(x)}{\partial x}$, *then output* $k^* = k - 1$, H_{k^*}, Γ_{k^*}, $\Delta_{k^*} = \alpha_{k^*}(\Gamma_{k^*}, x)$ *and stop the algorithm. Otherwise let* $H_k(x) = ((H_{k-1}(x))^{\mathrm{T}}, (F_k^4(x))^{\mathrm{T}})^{\mathrm{T}}$, $\Gamma_k = v_2$; *for* $k = 1$, *let* $\alpha_k(\Gamma_k, x) = \alpha(v_2, x)$; *for* $k \geqslant 2$, *let* $\alpha_k(\Gamma_k, x) = (\alpha(v_2, x)^{\mathrm{T}}, \alpha_{k-1}(\alpha(v_2, x), v_2, x)^{\mathrm{T}})^{\mathrm{T}}$; *and let* $\Delta_k = \alpha_k(\Gamma_k, x)$, $\widetilde{f}_k(x, \Gamma_k) := \widetilde{f}_{k-1}(x, v)|_{v_1 = \alpha(v_2, x)}$, $M_k = \{x : H_k(x) = 0\}$, $k := k + 1$, *and go to Step* (1).

Remark 3. By the output of Algorithm $1'$, one has $\Delta_{k^*} = \alpha_{k^*}(\Gamma_{k^*}, x)$, where Γ_{k^*} consists of some components of u. Therefore any friend $u \in \mathscr{F}(M^*)$, with a possible reordering of its components, can be parameterized as $u = ((\alpha_{k^*}(\Gamma_{k^*}, x))^{\mathrm{T}}, (\Gamma_{k^*})^{\mathrm{T}})^{\mathrm{T}} + \xi(x)$, where $\xi(x)$ is any function which vanishes on M^*.

3. Solvability of regulator equations

Let $x_a = (x^{\mathrm{T}}, w^{\mathrm{T}})^{\mathrm{T}}$, $F_a(x_a, u) = (F(x, w, u)^{\mathrm{T}}, s(w)^{\mathrm{T}})^{\mathrm{T}}$, then (1) is rewritten as $\dot{x}_a = F_a(x_a, u)$ and $e = h(x_a)$. Now apply Algorithm $1'$, one obtains the maximal output zeroing submanifold M^* in some neighborhood U of the origin. For simplicity, the neighborhood U is omitted. The manifold M^* can be written as the solution set of its defining equations: $M^* = M_{k^*} = \{x_a : H^*(x_a) := H_{k^*}(x_a) = 0\}^c$. Note that Algorithm $1'$ outputs the free input variable Γ_{k^*} and the function $\Delta_{k^*} = \alpha_{k^*}(\Gamma_{k^*}, x)$. Let rank $\frac{\partial H^*}{\partial x_a}$ be r locally, then by rearranging the components of x_a, one can suppose that $x_a = (x_{a1}, x_{a2})$, x_{a1} is r-dimensional, $H^*(x_{a1}, x_{a2}) = H^*(x_a)$, and there exists a function δ such that $H^*(\delta(x_{a2}), x_{a2}) \equiv 0$.

Proposition 1. *Suppose* M^* *is the maximal output zeroing submanifold for the system* (1), *and* M^* *is defined, in a small neighborhood* U *of the origin, as* $M^* = \{(x, w) \in U : \phi_1(x, w) = 0, \ldots, \phi_\ell(x, w) = 0\}$. *Then the regulator equation* (2), *and hence the output regulation problem, is solvable only if* $T^*M^* \cap \text{span}_{\mathscr{K}}\{dw\} = 0$ *when* x *is viewed as independent of* w, *where* $\text{span}_{\mathscr{K}}\{dw\} = \text{span}_{\mathscr{K}}\{dw_1, dw_2, \ldots, dw_q\}$, $T^*M^* = \text{span}_{\mathscr{K}}\{d\phi_1, d\phi_2, \ldots, d\phi_\ell\}$, *and* $\mathscr{K}$ *is the set of all the smooth functions in the variable* (x, w) *on* M^*.

Proof. Suppose the regulator equation (2) is solvable, then the solution $\pi(w)$ determines the set $N = \{(x, w) : x - \pi(w) = 0\}$ which is a controlled invariant submanifold of (1) contained in $\{(x, w) : h(x, w) = 0\}$. Thus $N \subseteq M^*$ holds locally, and it follows that $T^*M^* \subseteq T^*N$.

It is easy to verify that $T^*N = \text{span}_{\mathscr{K}}\{dx - \frac{\partial \pi(w)}{\partial w} dw\} = \text{span}_{\mathscr{K}}\{dx_1 - \frac{\partial \pi_1(w)}{\partial w} dw, \ldots, dx_n - \frac{\partial \pi_n(w)}{\partial w} dw\}$, and $T^*N \cap \text{span}_{\mathscr{K}}\{dw\} = 0$, therefore $T^*M^* \cap \text{span}_{\mathscr{K}}\{dw\} = 0$.

Since the solvability of the regulator equation (2) is necessary for the solvability of the output regulation problem, the condition $T^*M^* \cap \text{span}_{\mathscr{K}}\{dw\} = 0$ is also necessary to solve the output regulation problem. $\square$

Let $W := \text{span}_{\mathbb{R}}\{\tau_1, \ldots, \tau_k\}$ and V be linear subspaces of $\mathbb{R}^n$, $\dim W = k$, and $W \cap V = 0$, then $\dim \text{span}\{\tau_1 + \rho, \tau_2, \ldots, \tau_k\} = k$ for any $\rho \in V$. Thus, by applying Proposition 1 and some easy linear algebra, one has rank $\frac{\partial H^*}{\partial x} = \text{rank} \frac{\partial H^*}{\partial x_a} = r$. Therefore one can choose the vector x_{a1} so that neither of $\{w_1, w_2, \ldots, w_q\}$ is included. Thus $w_1, \ldots, w_q$ are contained in x_{a2}, and (x_{a1}, x_{a2}) can be written as (x_1, x_2, w), where $x_1 := x_{a1}, (x_2^{\mathrm{T}}, w^{\mathrm{T}})^{\mathrm{T}} := x_{a2}$. Now π can be partitioned into (π_1, π_2) in the same way as x, and $u(w)$ can be partitioned as $u(w) = (v^{\mathrm{T}}, (\alpha_{k^*}(v, \pi, w))^{\mathrm{T}})^{\mathrm{T}}$, where $v = \Gamma_{k^*}$. Partition F into $F = (F_1^{\mathrm{T}}, F_2^{\mathrm{T}})$ so that the first equation in the system (1) is written as $\dot{x}_1 = F_1(x_1, x_2, w, u)$ and $\dot{x}_2 = F_2(x_1, x_2, w, u)$. Since $\pi_1 = \delta(\pi_2, w)$ and $u = (v^{\mathrm{T}}, (\beta(v, \pi_2, w))^{\mathrm{T}})^{\mathrm{T}}$, where $\beta(v, \pi_2, w) = \alpha_{k^*}(v, \pi, w)|_{\pi_1 = \delta(\pi_2, w)}$, the differential equation in (2) can be written as

$$\frac{\partial \pi_2}{\partial w} s(w) = \widetilde{F}(\pi_2(w), w, v(w)), \tag{4}$$

where $\widetilde{F}(\pi_2(w), w, v(w)) := F_2(\delta(\pi_2, w), \pi_2, w, v(w), \beta(v(w), \pi_2(w), w))$. Now the following theorem is obtained by the above analysis.

Theorem 1. *There exists an open neighborhood* U *of the origin such that the following statements are equivalent*:

(1) *there exists a smooth function* $u = u(w)$ *such that the regulator equation* (2) *is solvable on* U;
(2) *there exists a smooth function* $v = v(w)$ *such that the center manifold equation* (4) *is solvable on* U.

4. Feasible controllers of the regulator equation

Proposition 2. *Let* $N_1 = \{x : \phi_1(x) = 0\}$, $N_2 = \{x : \phi_2(x) = 0\}$ *be two controlled invariant submanifolds of* (3)

on an open neighborhood U_0 of 0, where ϕ_1, ϕ_2 satisfy $\phi_1(x)|_{x=0} = 0, \phi_2(x)|_{x=0} = 0$. Then

(i) $N_1 \cap U_0 \supseteq N_2 \cap U_0$ holds if and only if $\frac{\partial \phi_1}{\partial x} f(x, u) = 0$ for all $u \in \mathscr{F}(N_1 \cap U_0)$ and $x \in N_2 \cap U_0$;

(ii) $\mathscr{F}(N_1 \cap U_0) \subseteq \mathscr{F}(N_2 \cap U_0)$ holds if and only if $\frac{\partial \phi_2}{\partial x} f(x, u) = 0$ for all $u \in \mathscr{F}(N_1 \cap U_0)$ and $x \in N_2 \cap U_0$.

Proof. (i) For any $u \in \mathscr{F}(N_1 \cap U_0)$, one has $\frac{\partial \phi_1}{\partial x} f(x, u)|_{N_1 \cap U_0} = 0$. Then $\frac{\partial \phi_1}{\partial x} f(x, u)|_{N_2 \cap U_0} = 0$ by $N_1 \cap U_0 \supseteq N_2 \cap U_0$, and the necessity holds. As for the sufficiency, it suffices to prove $\phi_1(x) = 0$ for all $x \in N_2 \cap U_0$. Note that $\dot{\phi}_1 = \frac{\partial \phi_1}{\partial x} f(x, u) = 0$ for $x \in N_2 \cap U_0$ and $u \in \mathscr{F}(N_1 \cap U_0)$. Now fix the conditions $x \in N_2 \cap U_0$ and $u \in \mathscr{F}(N_1 \cap U_0)$. Let Γ be any connected component of $N_2 \cap U_0$, and $x(t_0)$ any point in Γ for some t_0, then $\phi_1(x(t_0)) = 0$. This initial condition of the function $\phi_1(x(t))$ at $t = t_0$ and the equation $\dot{\phi}_1(x(t)) = 0$ determine a zero solution $\phi_1(x(t)) \equiv 0$ on Γ and hence on $N_2 \cap U_0$. Thus $N_1 \cap U_0 \supseteq N_2 \cap U_0$.

(ii) By the condition $\mathscr{F}(N_1 \cap U_0) \subseteq \mathscr{F}(N_2 \cap U_0)$, $\frac{\partial \phi_2}{\partial x} f(x, u)|_{N_2 \cap U_0} = 0$ for all $u \in \mathscr{F}(N_1 \cap U_0)$, thus the necessity follows. Now prove the sufficiency. For any $u \in \mathscr{F}(N_1 \cap U_0)$, one has $\frac{\partial \phi_2}{\partial x} f(x, u)|_{N_2 \cap U_0} = 0$, therefore $u \in \mathscr{F}(N_2 \cap U_0)$ and $\mathscr{F}(N_1 \cap U_0) \subseteq \mathscr{F}(N_2 \cap U_0)$ holds. $\square$

The following result is obvious (cf. Xia, 1993).

Proposition 3. *Under the same conditions of Proposition* 2, *and suppose, furthermore,* $x = ((x^1)^T, (x^2)^T)^T$, $x^2 = ((x^3)^T, (x^4)^T)^T$, $\phi_1 = x^1 - \alpha(x^2)$, $\phi_2 = ((\phi_1)^T, (\beta(x^2))^T)^T$, $\beta(x^2) = x^3 - \gamma(x^4)$, $\mathscr{F}(N_1 \cap U_0) \supseteq \mathscr{F}(N_2 \cap U_0)$, $W_1(x) := \frac{\partial \phi_1}{\partial x} f(x, u(x))$, $\widetilde{W}_1(x^2) := W_1(x)|_{x^1 = \alpha(x^2)}$, and $\frac{\partial \widetilde{W}_1}{\partial x^3} \not\equiv 0$ for some $u_0(x) \in \mathscr{F}(N_2 \cap U_0)$, then $N_1 \cap U_0 = N_2 \cap U_0$.*

Now consider the parametrization of the feasible controllers of the regulator equation (2). For simplicity, fix the notation N to be $\{x_a := (x^T, w^T)^T \in U_1:$ there exists a function $u(x)$ such that $x = \pi(w)$ solves the Eq. (2)$\}$. It is obvious that N is controlled invariant, and one needs only to parameterize $\mathscr{F}(N)$. Let $\phi(x_a) = x - \pi(w)$, then $\frac{\partial \phi}{\partial x_a}((F(x, w, u))^T, (s(w))^T)^T = 0$ holds on the set $\{x_a \in U_1: \phi = x - \pi(w) = 0, h(x, w) = 0\}$.

Definition 2. A submanifold C^* is called a minimal output zeroing control invariant submanifold for (3) on U_0 if it contains but not equals 0, and is defined by $\{x \in U_0: \Phi_*(x) = 0\}$, with $\Phi_*(x) = (\Phi_1, \Phi_2, \ldots, \Phi_\ell)^T$, such that C^* is controlled invariant on U_0 and the equality rank $((\frac{\partial \Phi_*}{\partial x})^T, (\frac{\partial \phi}{\partial x})^T)^T = $ rank $\frac{\partial \Phi_*}{\partial x}$ holds on $\{x \in U_0: \Phi_*(x) = 0\}$ for any function $\phi(x)$ satisfying $\phi(x)|_{x=0} = 0$, $\Omega := \{x \in U_0: \Phi_*(x) = 0, \phi(x) = 0\}$ is nonzero, and $\dot{\phi}(x(t)) = \frac{\partial \phi}{\partial x} f(x, u) = 0$ on Ω for all $u \in \mathscr{F}(C^* \cap U_0)$.

Proposition 4. *A manifold C^* is minimal output zeroing control invariant for the system* (3) *on an open neighborhood U_0*

of 0 if and only if it is a minimal submanifold in the set $\mathscr{T} := \{C: 0 \in C, C \neq 0, C \subseteq M^*, C$ is controlled invariant on $U_0, \mathscr{F}(C \cap U_0) \supseteq \mathscr{F}(M^* \cap U_0)\}$, where M^* is the maximal output zeroing submanifold of (3) on U_0 and M^* contains the origin.

Proof. Let $C^* = \{x: \Phi_*(x) = 0\}$ be a minimal output zeroing control manifold for (3) on U_0, now show that it is minimal in $\mathscr{T}$. It follows from Definition 2 that C^* is nonzero, contained in M^*, controlled invariant, and $\mathscr{F}(C^* \cap U_0) \supseteq \mathscr{F}(M^* \cap U_0)$, therefore $C^* \in \mathscr{T}$. If C^* is not minimal in $\mathscr{T}$, then there exists a manifold $W \in \mathscr{T}$ such that $W \subset C^*$, $W \neq C^*$, and W is defined by $\{x: \Phi_*(x) = 0, \Psi(x) = 0\}$. The function Ψ must satisfy $\dot{\Psi} = 0$ when restricted on W since W is controlled invariant. By the definition of C^*, the function Ψ must be algebraically dependent on Φ_* which results in that the set $\{x \in U_0: \Phi_*(x) = 0, \Psi(x) = 0\}$ equals the set $\{x \in U_0: \Phi_*(x) = 0\}$. This contradicts the hypothesis $W \neq C^*$, therefore C^* is minimal in $\mathscr{T}$.

Let $\tilde{C} = \{x \in U_0: \Psi(x) = 0\}$ be a minimal manifold in $\mathscr{T}$. For any $\phi(x)$ which satisfies $\phi(x)|_{x=0} = 0$, $\Omega := \{x \in U_0: \Psi(x) = 0, \phi(x) = 0\} \neq 0$, and $\dot{\phi}(x(t)) = \frac{\partial \phi}{\partial x} f(x, u) = 0$ when $u \in \mathscr{F}(\tilde{C} \cap U_0)$ and $x \in \Omega$, one has $\mathscr{F}(\Omega \cap U_0) \supseteq \mathscr{F}(\tilde{C} \cap U_0)$ by Proposition 2. Then it follows from the minimality of $\tilde{C}$ that $\Omega = \tilde{C}$. Therefore the function $\phi(x)$ must be algebraically dependent on $\Psi(x)$ on Ω. Now $\tilde{C}$ satisfies Definition 2, this ends the proof. $\square$

Algorithm 2. *Input the maximal output zeroing manifold $M^* = \{x \in U_0: H^*(x) = 0\}$ for the system* (3) *on U_0, where $H^*(0) = 0$, and output k^* and C_{k^*}.*

1. *Let $C_0 = M^*$, $S_0 = H^*$, $k = 1$.*
2. *Find if there is any function $\phi(x)$ such that $\phi(x)|_{x=0} = 0$, $\Omega := U_0 \cap C_{k-1} \cap \{x: \phi(x) = 0\}$ is nonzero, and $\dot{\phi}(x(t)) = \frac{\partial \phi}{\partial x} f(x, u) = 0$ for all $u \in \mathscr{F}(C_{k-1} \cap U_0)$ when x is restricted on Ω. If such a $\phi(x)$ exists, then let $S_k(x) = (S_{k-1}(x)^T, \phi(x))^T$, and $C_k = \{x \in U_0: S_k(x) = 0\}$.*

If rank $\frac{\partial S_k(x)}{\partial x} = $ rank $\frac{\partial S_{k-1}(x)}{\partial x}$ holds on $C_{k-1} \cap U_0$ or the above-mentioned $\phi(x)$ does not exist, then stop the algorithm and output $k^* = k - 1$ and C_{k^*}. Otherwise let $k := k + 1$ and repeat Step 2.

Proposition 5. *In the output of the above Algorithm* 2, *C_{k^*} is minimal output zeroing control invariant.*

Proof. Obviously C_{k^*} is nonzero and contained in M^*. It is also invariant with respect to $f(x, u_*(x))$ for any $u_* \in \mathscr{F}(C_{k^*} \cap U_0)$. For any function $\phi(x)$ such that $\phi(x)|_{x=0} = 0$, $\Omega := \{x \in U_0: S_{k^*}(x) = 0, \phi(x) = 0\}$ is nonzero, $\dot{\phi}(x(t)) = \frac{\partial \phi}{\partial x} f(x, u) = 0$ on Ω for all $u \in \mathscr{F}(C_{k^*} \cap U_0)$, by Step 2 of Algorithm 2 for step $k = k^* + 1$, such a ϕ satisfies rank $((\frac{\partial S_{k^*}}{\partial x})^T, (\frac{\partial \phi}{\partial x})^T)^T = $ rank $\frac{\partial S_{k^*}}{\partial x}$ on $\{x \in U_0: S_{k^*}(x) = 0\}$. Thus C_{k^*} is minimal output zeroing control invariant. $\square$

Definition 3. The system (3) is called left invertible on an open neighborhood U_0 of the origin if its maximal output zeroing submanifold M^* is defined by $M^* = \{x : H^*(x) = 0\}$ on U_0 with $H^*(0) = 0$, and it satisfies the following Left Invertibility Condition: the matrix $\frac{\partial H^*}{\partial x} \frac{\partial f(x,u)}{\partial u}$ is of full column rank, or equivalently left invertible, when x is restricted to $M^* \cap U_0$ and u is treated as a free variable.

Lemma 3. *Suppose* (3) *is left invertible on an open neighborhood U_0 of 0, and $M = \{x : H(x) = 0\}$ is a controlled invariant submanifold on U_0 with $H(0) = 0$, then*

(i) *the matrix $\frac{\partial H}{\partial x} \frac{\partial f(x,u)}{\partial u}$ is of full column rank on $M \cap U_0$ when u is treated as a free variable;*

(ii) *for any $u_0 \in \mathscr{F}(M \cap U_0)$, $\mathscr{F}(M \cap U_0)$ equals the set $\mathscr{S} := \{u = u_0 + \xi(x) : \xi(x) \text{ is any function which vanishes on } M \cap U_0\}$.*

Proof. (i) Let $M^* = \{x : H^*(x) = 0\}$ be the maximal output zeroing manifold on U_0 with $H^*(0) = 0$, then $M \subseteq M^*$ since M is controlled invariant. One can suppose that all the components of H^* appear in H. Then it follows from the left invertibility of $\frac{\partial H^*}{\partial x} \frac{\partial f(x,u)}{\partial u}$ on M^* that $\frac{\partial H}{\partial x} \frac{\partial f(x,u)}{\partial u}$ is also left invertible on M^* and hence on M, this proves (i).

(ii) For any $u_0 \in \mathscr{F}(M \cap U_0)$, one has $\frac{\partial H}{\partial x} f(x, u_0)|_{M \cap U_0} = 0$. Since the restriction of any $u = u_0 + \xi \in \mathscr{S}$ to $M \cap U_0$ still equals u_0, there are $\frac{\partial H}{\partial x} f(x, u)|_{M \cap U_0} = 0$, and $u = u_0 + \xi \in \mathscr{F}(M \cap U_0)$, $\mathscr{S} \subseteq \mathscr{F}(M \cap U_0)$.

On the other hand, for any element $u_1 \in \mathscr{F}(M \cap U_0)$, it suffices to show $u_1 - u_0$ vanishes on $M \cap U_0$ in order to prove $\mathscr{F}(M \cap U_0) \subseteq \mathscr{S}$. By the condition of left invertibility and the Inverse Function Theorem, u can be solved uniquely from the equation $\frac{\partial H}{\partial x} f(x, u)|_{M \cap U_0} = 0$, and one has $u = \alpha(x)$. Therefore, when restricted on $M \cap U_0$, both $u_0(x)$ and $u_1(x)$ have the same form $\alpha(x)$, and thus there exists a function $\xi(x)$, which vanishes on $M \cap U_0$, such that $u_1(x) = u_0(x) + \xi(x)$. Hence $\mathscr{F}(M \cap U_0) \subseteq \mathscr{S}$ and (ii) follows. $\square$

Theorem 2. *Let M^* be the maximal output zeroing manifold M^* of the system* (1) *on an open neighborhood U_0 of the origin with $0 \in M^*$, N the solution manifold of* (2) *on U_0, C^* the minimal output zeroing control invariant manifold of* (1) *on U_0, and $C^* \subseteq N$. Suppose* (1) *is left invertible on U_0, then both defining equations of N and C^* satisfy the Left Invertibility Condition in Definition 3, and therefore the equalities $\mathscr{F}(M^*) = \mathscr{S}_1$, $\mathscr{F}(N) = \mathscr{S}_2$, and $\mathscr{F}(C^*) = \mathscr{S}_3$ hold for any $u_0 \in \mathscr{F}(M^*)$, and $\mathscr{F}(C^*) \supseteq \mathscr{F}(N) \supseteq \mathscr{F}(M^*)$, where $\mathscr{S}_1 := \{u = u_0 + \xi(x) : \xi(x) \text{ is any smooth function which vanishes on } M^* \cap U_0\}$, $\mathscr{S}_2 := \{u = u_0 + \tau(x) : \tau(x) \text{ is any smooth function which vanishes on } N \cap U_0\}$, and $\mathscr{S}_3 := \{u = u_0 + \eta(x) : \eta(x) \text{ is any smooth function which vanishes on } C^* \cap U_0\}$.*

Proof. Let the defining equations of M^*, N, and C^* be H^*, Φ, and Ψ, respectively. By Lemma 3, Φ satisfies the Left Invertibility Condition, $\mathscr{F}(M^*) = \mathscr{S}_1$ and $\mathscr{F}(N) = \mathscr{S}_2$ hold. Note

that $C^* \subseteq N \subseteq M^*$, one can suppose Φ consists of some rows of Ψ. Since the matrix $\frac{\partial \Phi}{\partial x} \frac{\partial f(x,u)}{\partial u}$ is of full column rank on N, the matrix $\frac{\partial \Psi}{\partial x} \frac{\partial f(x,u)}{\partial u}$ must be of full column rank on N and hence on C^*. Again, by Lemma 3, Ψ satisfies the Left Invertibility Condition and $\mathscr{F}(C^*) = \mathscr{S}_3$ holds. $\square$

Remark 4. The solution manifold N may not be unique, thus a minimal output zeroing control invariant C^* may not be contained in a solution manifold N. There may be the case that the defining equations of C^* with respect to the system (1) contain a equation which has only the variable w and has no variable x. This does not affect the results of Theorem 2 since the conclusions about $F(C^*)$, $F(N)$, $F(M^*)$ and their relations still hold. It would be convenient to compute the minimal output zeroing control invariant manifold by noting that the function $\phi(x, w) = w$ always satisfies the equality $\dot{w} = 0$ when restricted on $\{(x, w) : w = 0\}$ (see C of Example 1).

By the proof of Theorem 2, one needs only to find a controlled invariant manifold contained in N in order to find $\mathscr{F}(N)$. Therefore Algorithm 3 may be stopped at an intermediate step as soon as one obtains a controlled invariant manifold contained in N (see C' of Example 1).

Proposition 6. *Assume that* (1) *is left invertible on an open set U_0, $0 \in U_0$, and $\Lambda = \{\alpha : N_\alpha \text{ is a solution manifold of} (2) \text{ on } U_0\}$. Suppose C^* is a minimal output zeroing manifold of* (1) *on U_0, $C^* \cap U_0 \subseteq (\bigcap_{\alpha \in \Lambda} N_\alpha) \cap U_0$, then $\mathscr{F}(C^* \cap U_0) \supseteq \bigcup_{\alpha \in \Lambda} \mathscr{F}(N_\alpha \cap U_0)$.*

The above proposition follows easily by using Lemma 3. The following example shows the computing steps for a general non-affine system. It is non-affine, therefore the results in Isidori and Bynes (1990) and Huang (2003) are not applicable. Again the reduction algorithm in Huang and Lin (1995) cannot be directly applied since the rank condition is not satisfied.

Example 1. Consider system (3) with $n = 6$, $f = (f_1, f_2, \ldots, f_6)^\mathrm{T}$, $f_1 = w_2^2 - w_1^2 + 2w_1 w_2 - 2x_1$, $f_2 = x_3$, $f_3 = -x_3 + x_6 + 2(x_4 + 1)u_2$, $f_4 = x_5 - (x_3 + 1)u_2^2$, $f_5 = -x_3 + x_4 + w_1 + u_1$, $f_6 = x_3 + w_2$, $\dot{w}_1 = w_2$, $\dot{w}_2 = -w_1$, $\dot{w}_3 = w_2$, $e_1 = h_1(x) = x_3$, $e_2 = h_2(x) = x_4$, and $e_3 = h_3(x) = w_3 - w_1 + x_3 - 2x_4$. Let $x_7 = w_1$, $x_8 = w_2$, $x_9 = w_3$, $H_0 = (h_1, h_2, h_3)^\mathrm{T}$, $x_a = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)^\mathrm{T}$, $F(x_a, u) = (f(x_a, u)^\mathrm{T}, w_2, -w_1, w_2)^\mathrm{T}$, $F_1 = \frac{\partial h}{\partial x_a} F(x_a, u) = (f_3, f_4, f_3 - 2f_4)^\mathrm{T}$. Then rank $\frac{\partial F_1}{\partial u} = 1$. Choose $F_1^1 = f_3$, $F_1^2 = f_4$, $v_1 = u_2$, $v_2 = u_1$ by Algorithm 1′. From $F_1^1 = 0$ one solves $u_2 = \frac{x_3 - x_6}{2(x_4 + 1)}$, substitute it into F_1^2 one has $F_1^4 = x_5 - \frac{(x_3 + 1)(x_3 - x_6)^2}{4(x_4 + 1)^2}$. Since F_1^4 is not a function of u, this convinces Lemma 2. Let $h_4 = F_1^4|_{h_1 = h_2 = h_3 = 0} = x_5 - \frac{x_6^2}{4}$ and $H_1 = (h_1, h_2, h_3, h_4)^\mathrm{T}$. Note that rank $\frac{\partial H_0}{\partial x_a} <$ rank $\frac{\partial H_1}{\partial x_a}$, one has $M_1 = \{x : H_1(x) = 0\}$. It is easy to find that Algorithm 1′ stops at $k = 1$ and $M^* = M_1$. The reduced regulator equation has a solution $\pi_1 = w_1 w_2$, $\pi_2 = w_1^2 + w_2^2$, $\pi_6 = w_1$. Therefore the original regulator equation is solvable when $u_1 = \frac{w_1 w_2}{2} - w_1$ and $u_2 = \frac{-\pi_6}{2} = \frac{-w_1}{2}$, and the solution is: $\pi_1 = w_1 w_2$, $\pi_2 = w_1^2 + w_2^2$,

$\pi_3 = \pi_4 = 0$, $\pi_5 = \frac{w_1^2}{4}$, $\pi_6 = w_1$. Denote the corresponding solution manifold by N. A simple calculation shows that the original system is left invertible. By Algorithm 2, $C = \{x_a : x_1 = x_2 = x_3 = x_4 = 0, x_5 - \frac{x_6^2}{4} = 0, w_1 = w_2 = w_3 = 0\}$ is minimal and contained in N, therefore $\mathscr{F}(C \cap U_0) \supseteq \mathscr{F}(M^* \cap U_0)$. However, by Remark 4, $C' = \{x_a : x_1 = x_2 = x_3 = x_4 = 0, x_5 - \frac{x_6^2}{4} = 0\}$ is controlled invariant and $C' \cap U_0 \subseteq N \cap U_0$, therefore $\mathscr{F}(C' \cap U_0) \supseteq \mathscr{F}(M^* \cap U_0)$, and it is finer than $\mathscr{F}(C \cap U_0) \supseteq \mathscr{F}(M^* \cap U_0)$ since $C \subseteq C'$ and $\mathscr{F}(C \cap U_0) \supseteq \mathscr{F}(C' \cap U_0)$.

5. Conclusion

In this paper a geometric characterization on the solvability of regulator equation is presented. A generalized algorithm for the computation of maximal output zeroing submanifold as well as its application in the simplification of regulator equations are obtained. The whole setup does not rely on the existence of relative degrees. A necessary condition for the solvability of output regulation problem is also obtained as a byproduct. By introducing the concepts of minimal output zeroing control invariant manifold and left invertibility, the parametrization of the friend set of the solution manifold of regulator equation is discussed too. For a future study, the challenging problem to find finer characterization of the set of feasible controllers is left to interested readers.

Acknowledgments

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