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On delta-modulated control: A simple system with complex dynamics

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Abstract

In this paper, we investigate some interesting properties of a scalar system controlled by Δ-modulated feedback. We show that there are three different cases. In the first case, there is a minimal global attractor which consists of only two points. The two points form either one 2-periodic orbit or two 1-periodic orbits (fixed points). We also characterize the attracting region for each of these two points. In the second case, the maximal stabilizable region is bounded, and there is a minimal local attractor inside this stabilizable region. In the third case, the maximal stabilizable set is a Cantor set, which is a repeller of the system, and the system is chaotic on the Cantor set.

1. Introduction

We consider a discrete-time scalar system

$$x^{+} = ax + u, \tag{1}$$

where x^{+} denotes the system state at the next discrete-time, a is a real number, and the control u is only allowed to take the value Δ or $-\Delta$ for a constant $\Delta > 0$. The control objective is to stabilize the system in the sense of driving the system state as close as possible to $x = 0$.

A practical example of this kind of control is the transmitting power control of a mobile unit in the direct sequence code division multiple access (DS-CDMA) cellular network. A simplest model is

$$x^{+} = x + u, \quad u = -\Delta \operatorname{sgn} x,$$

where x is the error of the mobile unit's power level received at the base station with respect to the desired value (both in decibel, dB). The control action stems from a simple idea: when the level of the received power is higher than the desired

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level, it is decreased by Δ dB, and when the level of the received power is lower than the desired level, it is increased by the same amount [1]. There is only one design parameter, Δ, and the power increment is either Δ or $-\Delta$. This scheme is called a delta modulation or DM transmitting power control. An advantage of such a control is that Δ can be stored at the base station or the active mobile unit, and the base station only needs to send 1 or -1 to command the increase or decrease of the power level. In other words, only one bit of datum is necessary for the implementation of the DM control. The requirement of one bit for transmitting power control is the standard of IS-95 [11].

In electronic circuits, such a method of analog-to-digital conversion is also called sigma–delta ($\Sigma\Delta$) modulation which was introduced much earlier in [9,4] and studied in depth for the "leaky" case (when $a \neq 1$) in a number of later publications [10,8,12]. Delta-modulated control is a special kind of *quantized control*, a topic of recently renewed interest [5,3,6]. Delta-modulated control is a two-level quantized control, and a quantized control is a cascade of delta-modulated controls. The study of delta-modulated control will eventually be helpful in the implementation of a quantized control. On the other hand, the recent interest in quantized control has been focused on the design of quantization levels for the purpose of stabilization.

In this paper, we present some new results. It will be proved below that for system (1) under delta-modulated feedback:

- when $|a| \leqslant 1$,
  - there is a minimal global attractor,
  - when $|a| = 1$, the attractor is a bounded interval,
  - when $|a| < 1$, it consists of only two points, and in the later case, we can characterize the attracting region for each of the two points; this result, reported in [15] with a different proof, is included here for completeness.
- when $1 < |a| \leqslant 2$,
  - there is a maximal stabilizable region,
  - in the above stabilizable region, there is a local attractor, the interval $[-\Delta, \Delta]$, which is independent of a;
- when $|a| > 2$,
  - the maximal stabilizable set is a Cantor set,
  - this Cantor set is a repeller of system (1),
  - system (1) is chaotic on this Cantor set under delta-modulated control.

The mathematical terms in the above will be made precise throughout the development below. Comparing with the existing results, the new contributions of this paper consist of the construction of the Cantor-set repeller (when $|a| > 2$) as well as a mathematically elegant method of proving the chaotic nature of this repeller.

The rest of the paper will be devoted to the relevant constructions and proofs of the aforementioned properties. We will use approaches of different complexity to deal with the three cases: a direct method for the case of $|a| \leqslant 1$, a Lyapunov function approach for the case of $1 < |a| \leqslant 2$, and finally, for the case of $|a| > 2$, a more mathematically sophisticated approach based on fractal geometry. We will proceed in such a way that a possible generalization to higher-dimensional cases can be carried out in the near future.

2. Delta-modulated control

First, we review some concepts from dynamical systems theory.

Let D be a subset of $\mathbb{R}^n$, and assume that $f: D \to D$ is a mapping. The following equation describes a dynamical system defined by f on D:

$$x^+ = f(x). \tag{2}$$

An orbit of (2) starting from x is denoted as $\{f^k(x),\ k \geqslant 0\}$, where f^k represents the kth iteration of $f: f^0(x) = x$, $f^1(x) = f(x)$, $f^2(x) = f(f(x)), \ldots$ x is called a periodic point of f if there is a positive integer p such that $f^p(x) = x$, and the smallest such integer p is called the (prime) period of x.

A subset F of D is an attractor of f, if F is closed, invariant with respect to f (i.e., $f(F) = F$), and attractive (i.e., there is an open set V containing F such that $f^k(x) \to F$ for all $x \in V$). The set V is called the attracting region of F. If $V = D$, then F is called a global attractor of f on D.

Similarly, a closed invariant set F is called a repeller of f if, starting from any x outside of F, the orbit of (2) moves away from F.

2.1. Case: $|a| \leqslant 1$

Theorem 1. *When $|a| \leqslant 1$, there exists a delta-modulated feedback control such that*

(1) *when $|a| = 1$, $\Omega = [-\Delta, \Delta]$ is a global attractor on $(-\infty, \infty)$;*
(2) *when $|a| < 1$, the global attractor is the following set of two points:*

$$\{-\Delta/(1+|a|), \Delta/(1+|a|)\}; \tag{3}$$

(3) *when $0 \leqslant a < 1$, the two points in (3) are 2-periodic points; when $-1 < a < 0$, the two points in (3) are (1-periodic) fixed points.*

Proof. We define the delta-modulated feedback as

$$u = -\Delta \mathrm{sgn}(ax),$$

in which $\mathrm{sgn}(x)$ is a function satisfying

$$\mathrm{sgn}(x) = \begin{cases} 1, & \text{when } x \geqslant 0; \\ -1, & \text{when } x < 0. \end{cases}$$

With such a feedback, the closed-loop system is

$$x^+ = ax - \Delta \mathrm{sgn}(ax) \stackrel{\text{def}}{=} f_c(x). \tag{4}$$

The right-hand side of (4) is well-defined but discontinuous. Systems like this can be studied by the theory of difference inclusion. Another approach is to apply probability as was done in [2]. We take a deterministic approach to ensure the uniqueness of an orbit.

In the following, we only give a proof to the case when $0 \leqslant a \leqslant 1$. A proof for the case of $-1 \leqslant a < 0$ can be worked out in similar lines.

Our proof is divided into five parts.

1. For $0 \leqslant a \leqslant 1$, $f_c(\Omega) \subset \Omega$.
   (a) When $0 \leqslant x \leqslant \Delta$,

$$x^+ = ax - \Delta \geqslant -\Delta.$$

Also

$$x^+ = ax - \Delta \leqslant ax \leqslant x \leqslant \Delta.$$

   (b) When $-\Delta \leqslant x \leqslant 0$,

$$x^+ = ax + \Delta \leqslant \Delta.$$

Also

$$x^+ = ax + \Delta \geqslant ax \geqslant -a\Delta \geqslant -\Delta.$$

2. For $0 \leqslant a \leqslant 1$, Ω is globally attractive.
   (a) If $x > \Delta$, then

$$x^+ = ax - \Delta \leqslant x - \Delta.$$

So $f_c^{k+1}(x) = x^+ \leqslant x = f_c^k(x)$, since $f_c^k(x) = x > \Delta$ is positive, namely, $f_c^k(x)$ is decreasing as long as it is positive. We prove, by contradiction, that $\{f_c^k(x)\}$ enters Ω. Suppose this is not the case. Then there are only two situations:

Case (i): $f_c^k(x) > \Delta$ for all k.

Case (ii): There exists a positive integer l such that

$$x > f_c(x) > \cdots > f_c^l(x) > \Delta,$$

but

$$f_c^{l+1}(x) < -\Delta.$$

In case (i), since $\{f_c^k(x)\}$ is decreasing and bounded from below, we have

$$f_c^k(x) \to x^* \geqslant \Delta$$

and, therefore,

$$x^* = ax^* - \Delta$$

or

$$(1-a)x^* = -\Delta,$$

which is impossible, since $(1-a)x^*$ is non-negative when $0 \leqslant a \leqslant 1$ and $x^* \geqslant \Delta$.
In case (ii), by assumption we have

$$f_c^{l+1}(x) = af_c^l(x) - \Delta < -\Delta,$$

hence $af_c^l(x) < 0$, which is impossible since $a \geqslant 0$ and $f_c^l(x) > \Delta$.

(b) If $x \leqslant -\Delta$, then

$$x^+ = ax + \Delta \geqslant x + \Delta.$$

So $f_c^{k+1}(x) \geqslant f_c^k(x)$, if $f_c^k(x)$ is negative. Similarly, we can prove that $\{f_c^k(x)\}$ enters Ω.

3. When $a = 1$, Ω is an attractor.
   From part 1 of the proof, $f_c(\Omega) \subset \Omega$. We only need to prove that $\Omega \subset f_c(\Omega)$.
   To see this, first note that $0 \in f_c(\Omega)$ since $f_c(\Delta) = 0$.
   For any $0 \neq y \in \Omega$, define

$$\bar{x} = y - \Delta \mathrm{sgn} y.$$

Then, since $f_c(\Omega) \subset \Omega$, we have $\bar{x} \in \Omega$. Note that $\bar{x}$ and y have opposite signs (e.g., if $y > 0$, then since $0 < y \leqslant \Delta$, we have $\bar{x} = y - \Delta < 0$), so

$$f_c(\bar{x}) = y.$$

From this last equation, we have $f_c^2(y) = y$. We conclude that (when $a = 1$): any point in the half open interval $(-\Delta, \Delta]$ is a 2-periodic point.
It is also straightforward to verify that when $a = -1$: (i) all points but $\pm\Delta/2$ in the closed interval $[-\Delta, \Delta]$ are 2-periodic; (ii) $\pm\Delta/2$ are fixed points.

4. When $0 \leqslant a < 1$, the attractor is $\{-\Delta/(1+|a|), \Delta/(1+|a|)\}$, which is a 2-periodic orbit.
   From the above proof, the attractor, if exists, belongs to $\Omega = [-\Delta, \Delta]$. It is therefore interesting to see how f_c evolves on Ω. Note that f_c transforms Ω into

$$f_c(\Omega) = [-\Delta, -(1-a)\Delta] \cup [(1-a)\Delta, \Delta],$$

therefore $(-(1-a)\Delta, (1-a)\Delta)$ is cut away, and it does not belong to the f_c-invariant set in Ω.
One step further, we can also show that

$$f_c^2(\Omega) = f_c([-\Delta, -(1-a)\Delta] \cup [(1-a)\Delta, \Delta]) = [-(1-a+a^2)\Delta, -(1-a)\Delta] \cup [(1-a)\Delta, (1-a+a^2)\Delta].$$

Generally, if we denote

$$f_c^k(\Omega) = [-a_k, -b_k] \cup [b_k, a_k]$$

for $a_k \geqslant b_k > 0$, then we can easily show that

$$f_c^{k+1}(\Omega) = [ab_k - \Delta, aa_k - \Delta] \cup [\Delta - aa_k, \Delta - ab_k],$$

that is,

$$\begin{aligned} a_{k+1} &= \Delta - ab_k, \\ b_{k+1} &= \Delta - aa_k. \end{aligned} \tag{5}$$

For example, we have

$$a_0 = \Delta, \quad b_0 = 0, \quad a_1 = \Delta, \quad b_1 = (1-a)\Delta, \quad a_2 = (1-a+a^2)\Delta, \quad b_2 = (1-a)\Delta.$$

From Eqs. (5), we have

$$a_{k+2} = (1-a)\Delta + a^2 a_k, \tag{6}$$
$$b_{k+2} = (1-a)\Delta + a^2 b_k. \tag{7}$$

We can prove, by mathematical induction, that

(1) $a_{2i+1} = a_{2i}$, for $i = 0, 1, 2, \ldots$;
(2) $\{a_k\}$ is a decreasing sequence.

To prove (1), we first notice that $a_1 = a_0 = \Delta$. Suppose $a_{2i+1} = a_{2i}$. Then, from (6),

$$a_{2i+3} - a_{2i+2} = a^2(a_{2i+1} - a_{2i}) = 0.$$

To prove (2), observe that

$$a_2 = \Delta - a\Delta + a^2\Delta = \Delta - a\Delta(1-a) \leqslant \Delta = a_1.$$

Suppose $a_{2i} \leqslant a_{2i-1}$. Then, from (6) and part (1),

$$a_{2i+2} - a_{2i+1} = a^2(a_{2i} - a_{2i-2}) = a^2(a_{2i} - a_{2i-1}) \leqslant 0.$$

Combining part (1) with this fact, we have proved that $\{a_k\}$ is decreasing.
In completely similar lines, we can prove that $b_{2i-1} = b_{2i}$ and $\{b_k\}$ is increasing.
Therefore, since $a_k \geqslant b_k > 0$, there exist $a^* \geqslant b^* > 0$ such that as $k \to \infty$,

$$a_k \to a^*,$$
$$b_k \to b^*.$$

From (6) and (7), respectively, we find that $a^* = b^* = \Delta/(1+a)$. In other words,

$$\bigcap_{k=1}^{\infty} f_c^k[-\Delta, \Delta] = \{-\Delta/(1+a), \Delta/(1+a)\},$$

which, hence, is the global attractor.It can be easily verified that $\{-\Delta/(1+a), \Delta/(1+a)\}$ is the 2-periodic orbit of the closed-loop system.

5. When $-1 < a < 0$, it can be similarly verified that $\{-\Delta/(1-a), \Delta/(1-a)\}$ is a global attractor, and these two points are (1-periodic) fixed points of the closed-loop system. □

Remark 1. Note that when $0 \leqslant a < 1$, we have a situation where there are 2-periodic points but no 1-periodic points for a mapping on $[-\Delta, \Delta]$. This is in sharp contrast with the continuous case covered by the Sarkovskii theorem [13].

Since the periodic points are globally attractive, it is interesting to find out the attracting region for each of the periodic points.

First, we introduce a new concept. For any real number x and $a \neq 0$ (the case $a = 0$ is trivial), the characteristic index κ is defined as the following non-negative integer:

$$\kappa = \left\lfloor \log_{|a|}\left(\frac{\Delta}{\Delta + (1-|a|)|x|}\right)\right\rfloor,$$

where $\lfloor * \rfloor$ denotes the floor, i.e., the maximal integer bounded by the real number $*$.

Lemma 1

(i) *For any x, the characteristic index κ is the smallest non-negative integer m such that*

$$|f_c^{(m)}| < \frac{\Delta}{|a|}.$$

(ii.1) *For* $-1 < a < 0$, κ *is the smallest non-negative integer* m *such that* $f_c^{(m)}$ *and* $f_c^{(m+1)}$ *have the same sign*;
(ii.2) *For* $0 < a < 1$, κ *is the smallest non-negative integer* m *such that* $f_c^{(m)}$ *and* $f_c^{(m+1)}$ *have opposite signs.*

Proof. We prove the result only for the case when $0 < a < 1$. Proof for another case can be worked out in similar lines, and it is therefore omitted.

If $0 < a < 1$, it follows that

$$f_c^{(m+1)} = af_c^{(m)} - \operatorname{sgn}(f_c^{(m)})\Delta = \begin{cases} af_c^{(m)} - \Delta, & f_c^{(m)} \geqslant 0; \\ af_c^{(m)} + \Delta, & f_c^{(m)} < 0. \end{cases} \tag{8}$$

It is easy to see that $|f_c^{(m)}| < \Delta/a$ if and only if $f_c^{(m)}$ and $f_c^{(m+1)}$ have different signs.

Note that for $m \leqslant \kappa$:

- if $x > 0$, then

$$\begin{aligned} f_c(x) &= ax - \Delta, \\ f_c^{(2)}(x) &= af_c(x) - \Delta \\ &= a^2x - a\Delta - \Delta, \\ &\vdots \\ f_i^{(m)}(x) &= a^mx - a^{m-1}\Delta - \cdots - a\Delta - \Delta \\ &= a^mx - \frac{(1-a^m)}{(1-a)}\Delta \\ &= a^m|x| - \frac{(1-a^m)}{(1-a)}\Delta; \end{aligned}$$

- if $x \leqslant 0$, then

$$\begin{aligned} f_c(x) &= ax + \Delta, \\ f_x^{(2)}(x) &= af_c(x) + \Delta \\ &= a^2x + a\Delta + \Delta, \\ &\vdots \\ f_c^{(m)}(x) &= a^mx + a^{m-1}\Delta + \cdots + a\Delta + \Delta \\ &= a^mx + \frac{(1-a^m)}{(1-a)}\Delta \\ &= -a^m|x| + \frac{(1-a^m)}{(1-a)}\Delta. \end{aligned}$$

It is then straightforward to verify that the real number $s = \log_a \frac{\Delta}{\Delta + (1-a)|x|}$ satisfies

$$a^s|x| - \frac{(1-a^s)}{(1-a)}\Delta = 0.$$

Therefore, $\kappa = \lfloor s \rfloor$ is the smallest integer such that $f_c^{(m)}$ changes sign.

This completes the proof of the lemma. □

The analysis given in the proof can be useful in finding the limiting periodic points. We will carry out this separately for the two types of systems with $-1 < a < 0$ and $0 < a < 1$, respectively.

If $-1 < a < 0$, then we have

$$f_c^{(m+1)}(x) = f_c(f_c^{(m)})(x) = af_c^{(m)}(x) + \operatorname{sgn}(f_c^{(m)}(x))\Delta.$$

By (ii.1) of Lemma 1, $f_c^{(m)}$ has the same sign as $f_c^{(\kappa)}$, for $m \geqslant \kappa$. Therefore, we have, for $m \geqslant \kappa$,

$$f_c^{(m+1)}(x) = af_c^{(m)}(x) + \operatorname{sgn}(f_c^{(\kappa)}(x))\Delta.$$

Hence, by denoting the limit of $f_c^{(m)}$ by x^*, we can solve x^* from

$$x^* = ax^* + \mathrm{sgn}(f_c^{(\kappa)})\Delta,$$

to obtain

$$x^* = \frac{\mathrm{sgn}(f_c^{(\kappa)})\Delta}{1-a}.$$

If $0 < a < 1$, then first let κ_e be the next even integer (or zero) after κ (that is, $\kappa_e = \kappa$ if κ is even or zero, and $\kappa_e = \kappa + 1$ if κ is odd). Then, from (ii.2) of Lemma 1, $f_c^{(2m)}$ have the same sign as $f_c^{(\kappa_e)}$, for $m \geqslant \frac{\kappa_e}{2}$. Therefore, we have, for $2m \geqslant \kappa_e$,

$$f_c^{(2(m+1))} = a^2 f_c^{(m)} - a\mathrm{sgn}(f_c^{(\kappa_e)})\Delta + \mathrm{sgn}(f_c^{(\kappa_e)})\Delta.$$

Hence, if denoting the limit of $f_c^{(2m)}$ by x^*, then we can solve x^* from

$$x^* = a^2 x^* - a\mathrm{sgn}(f_c^{(\kappa_e)})\Delta + \mathrm{sgn}(f_c^{(\kappa_e)})\Delta,$$

to obtain

$$x^* = \frac{\mathrm{sgn}(f_c^{(\kappa_e)})\Delta}{1+a}.$$

Summarizing the above development, we have the following characterization of the attracting region of a periodic point.

Theorem 2. *For any x, denote its characteristic index as* κ.

(i) *For* $-1 < a < 0$, *x belongs to the attracting region of* $\frac{\Delta}{1-a}\left(-\frac{\Delta}{1-a}\right)$ *if and only if* $sgn(x^{(\kappa)}) = 1(sgn(x^{(\kappa)}) = -1)$.

(ii) *For* $0 \leqslant a < 1$, *x belongs to the attracting region of* $\frac{\Delta}{1+a}\left(-\frac{\Delta}{1+a}\right)$ *if and only if* $sgn(x^{(\kappa)}) = (-1)^{\kappa}(sgn(x^{(\kappa)}) = (-1)^{\kappa+1})$.

2.2. Case: $1 < |a| \leqslant 2$

In this section, our approach is based on the Lyapunov function

$$V(x) = x^2.$$

The orbit of system (1) comes closer to the origin if $V(x)$ decreases along the system's orbits. The increment of $V(x)$ along the orbit of system (1) is

$$V_\Delta(x) \stackrel{\mathrm{def}}{=} V(x^+) - V(x) = (a^2-1)x^2 + 2axu + u^2 = (u - u^{(1)}(x))(u - u^{(2)}(x)), \tag{9}$$

where

$$u^{(1)}(x) = -ax - |x|,$$
$$u^{(2)}(x) = -ax + |x|.$$

For any $x \neq 0$, the set defined by

$$U(x) = \{u \in \mathbb{R} | u^{(1)} < u < u^{(2)}\}$$

is the control set that makes $V(x)$ decreasing.

Note that when the open-loop system is stable, i.e., when $|a| \leqslant 1$, $u^{(1)}(x)$ and $u^{(2)}(x)$ do not have the same sign for any $x \neq 0$. When $|a| > 1$, $u^{(1)}(x)$ and $u^{(2)}(x)$ have the same sign. Generally, $u^{(1)}(x)$ and $u^{(2)}(x)$ define, on the (x, u) plane, a radiative cone, and the Lyapunov function can be made negative with control values falling inside the cone. This interpretation is depicted in Fig. 1.

The following intuitive approach can be modified to simplify the development of the previous subsection for the case of $|a| \leqslant 1$, therefore from now on we only consider the case of $|a| > 1$. Define

$$\Gamma = \left(-\frac{\Delta}{|a|-1}, \frac{\Delta}{|a|-1}\right),$$

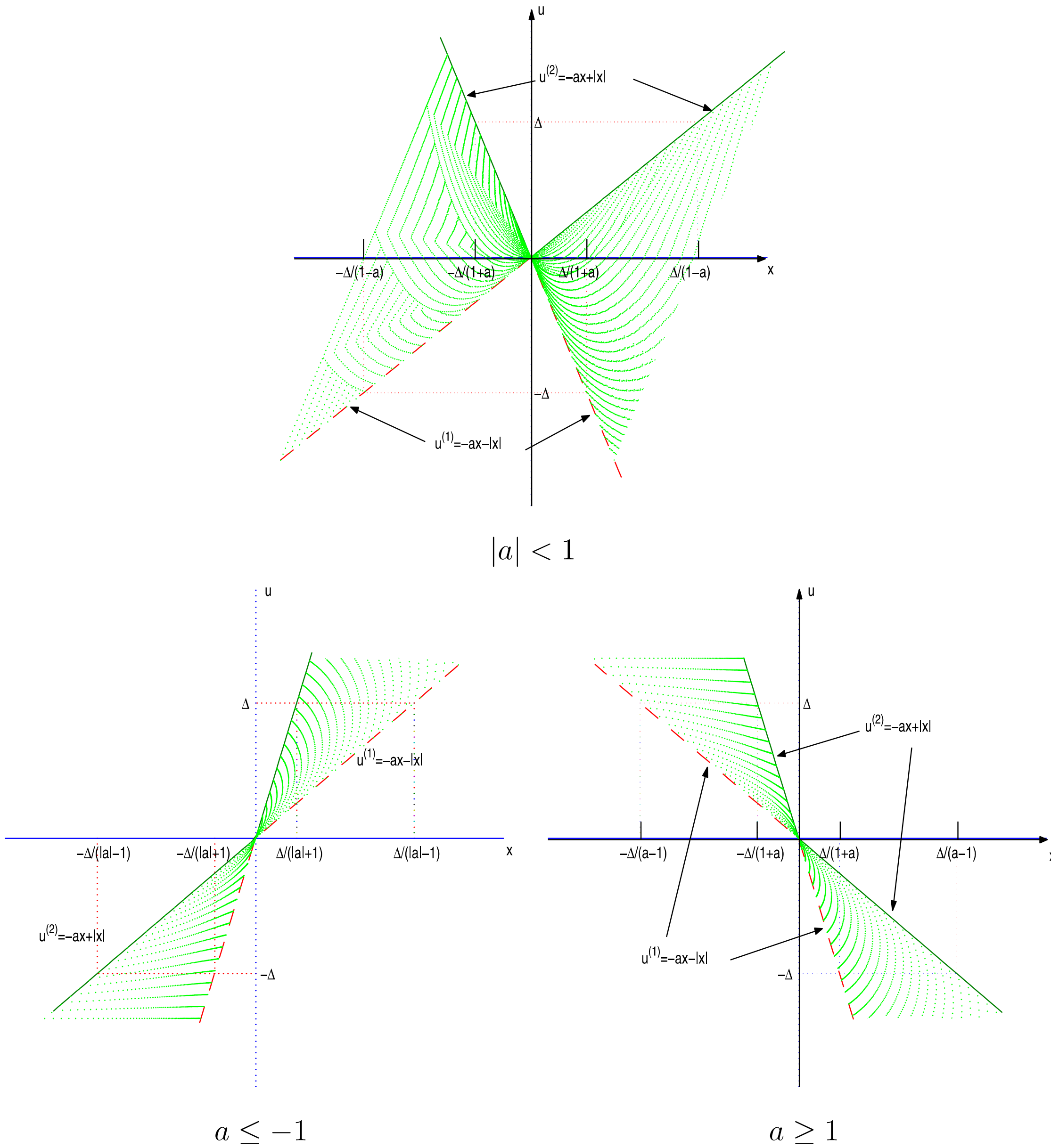

Fig. 1. The cone-shaped regions of the control Lyapunov function.

and

$$\Omega_c = \left[-\frac{\Delta}{|a|+1}, \frac{\Delta}{|a|+1}\right].$$

Also can be seen from Fig. 1 is that, on $\mathbb{R}\setminus\Gamma = \left(-\infty - \frac{\Delta}{|a|-1}\right] \cup \left[\frac{\Delta}{|a|-1}, \infty\right)$, $V(x)$ cannot be made decreasing with a bounded control: $|u| \leqslant \Delta$. Consequently, $V(x)$ cannot be decreased by Δ-modulated control.

Theorem 3

(i) *When* $|a| > 1$, *any orbit of* (1) *with any bounded control* $u \in [-\varDelta, \varDelta]$ *starting from outside* Γ *moves away from* Γ.
(ii) *When* $|a| = 2$, *there exists a* $\varDelta$*-modulated feedback such that* $\Omega = \Gamma = [-\varDelta, \varDelta]$ *is the minimal invariant set with respect to the closed-loop system.*
(iii) *When* $1 < |a| < 2$, *there exists a* $\varDelta$*-modulated feedback such that* $\Omega = [-\varDelta, \varDelta]$ *is the attractor of the closed-loop system, with* Γ *as the attracting region. Therefore,* Γ *is the stabilizable region and* Ω *is the attractor of the closed-loop system.*

Proof

(i) Note that $x \in \mathbb{R} \setminus \Gamma$ if and only if

$$V(x) \geqslant \frac{\varDelta^2}{(|a|-1)^2}.$$

Also, for $x \in \mathbb{R} \setminus \Gamma$, we either have

$$u^{(2)}(x) > u^{(1)}(x) \geqslant \varDelta$$

or have

$$u^{(1)}(x) < u^{(2)}(x) \leqslant -\varDelta.$$

So, for any control input u belonging to $U = [-\varDelta, \varDelta]$,

$$V_\varDelta(x) = (u - u^{(1)}(x))(u - u^{(2)}(x)) \geqslant 0,$$

that is,

$$V(x^+) \geqslant V(x) \geqslant \frac{\varDelta^2}{(|a|-1)^2}.$$

Consequently, $x^+ \in \mathbb{R} \setminus \Gamma$, namely, x^+ moves away from Γ.

(ii) This case is listed separately due to notational conformity. The proof is actually contained in the following proof for (iii).

(iii) For any $x \in \Gamma \backslash \Omega_c$, $\varDelta$ or $-\varDelta$ falls into between $u^{(1)}(x)$ and $u^{(2)}(x)$. It can be easily verified that

$$u = -\varDelta \operatorname{sgn}(ax) \tag{10}$$

is a $\varDelta$-modulated feedback to make $V(x)$ decreasing for $x \in \Gamma \backslash \Omega_c$, and to make $V(x)$ increasing for $x \in \Omega_c$.
Note that $\Omega_c \subset \Omega$ and when $|a| < 2$, $\Omega \subset \Gamma$.
The proof then splits into two parts:

(iii.1) Ω is invariant with respect to f_c.For any $x \in \Omega \backslash \Omega_c$,

$$V(x^+) < V(x) \leqslant \max_{x \in \Omega \backslash \Omega_c} V(x) = \varDelta^2.$$

For any $x \in \Omega_c$, since

$$V(x^+) = a^2x^2 - 2\varDelta|ax| + \varDelta^2,$$

the maximal value of $V(x^+)$ on Ω_c is reached at $x = 0$ (see Fig. 2), therefore,

$$V(x^+) \leqslant \max_{x \in \Omega_c} V(x^+) = \varDelta^2.$$

So, in both cases, $x^+ \in \Omega$, showing that $f_c(\Omega) \subset \Omega$.We can also show that $\Omega \subset f_c(\Omega)$, as follows.
For any $y \in \Omega$, choose

$$\bar{x} = \frac{y - \operatorname{sgn}(y)\varDelta}{a}.$$

Then, it verifies that

$$|\bar{x}| \leqslant |y - \operatorname{sgn}(y)\varDelta| = ||y| - \varDelta| \leqslant \varDelta,$$

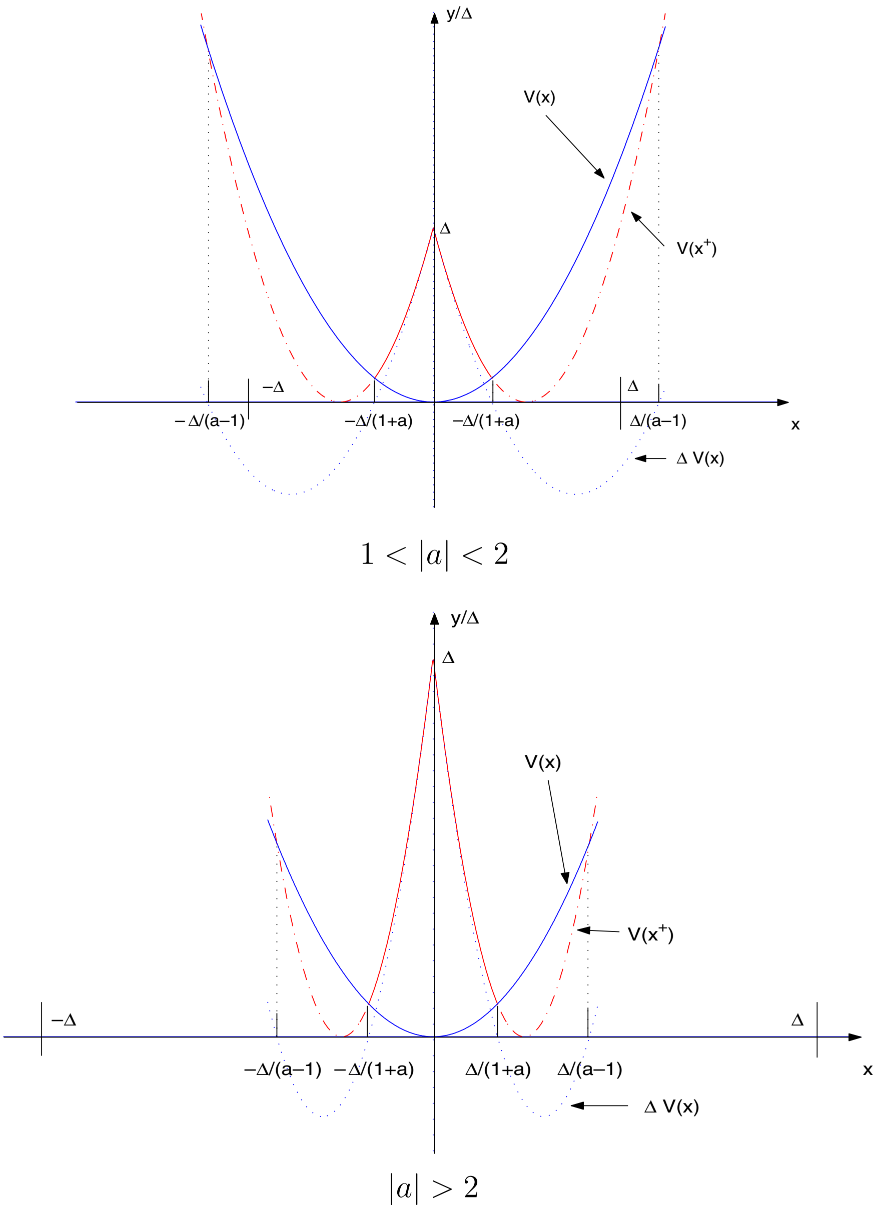

Fig. 2. The control Lyapunov function: its maximal value and variational difference.

i.e., $\bar{x} \in \Omega$, and

$$f_c(\bar{x}) = y.$$

Hence, $f_c(\Omega) = \Omega$. So, Ω is invariant with respect to f_c.

(iii.2) The attracting region of Ω is Γ.For any $x \in \Gamma \backslash \Omega$, denoting $x_0 = x$, $x_1 = f_c(x)$, and $x_k = f_c(x_{k-1})$, for $k = 1, 2, \ldots$, there can only be two cases:

(1) there is an integer N such at $x_N \in \Omega$;

(2) for any k, $x_k \notin \Omega$.

We next show that the second case will never happen. To see this, $V(x)$ is, by definitions of Γ and Ω, decreasing. So, $x_k \in \Gamma$. Therefore, we conclude that the sequence $\{V(x_k)\}$ is a decreasing sequence satisfying

$$\Delta^2 < V(x_k) < \left(\frac{\Delta}{1-|a|}\right)^2.$$

So there is a limit, denoted by V^*, of $\{V(x_k)\}$ as $k \to \infty$, satisfying

$$\Delta^2 \leqslant V^* < \left(\frac{\Delta}{1-|a|}\right)^2. \tag{11}$$

Note that

$$V(x_{k+1}) = a^2 V(x_k) - 2\Delta|a|\sqrt{V(x_k)} + \Delta^2.$$

Taking the limit as $k \to \infty$, V^* can be solved from

$$V^* = a^2 V^* - 2\Delta|a|\sqrt{(V^*)} + \Delta^2,$$

as

$$V^* = \left(\frac{\Delta}{|a| \pm 1}\right)^2.$$

This is a contradiction to (11).

This proves that $\{x_k\}$ eventually enters Ω for any $x \in \Gamma\backslash\Omega$, and thus Γ is contained in the attracting region of Ω. □

Combining this with the conclusion in (i), the attracting region of Ω is Γ.

2.3. Case: $|a| > 2$

From Theorem 3, with the Δ-modulated feedback (10), any point in

$$\mathbb{R} \setminus \Gamma = \left\{x \middle| |x| \geqslant \frac{\Delta}{|a|-1}\right\}$$

cannot be driven to the inside of interval Γ.

Here, we investigate what happens inside the closed interval $\overline{\Gamma} = \left[-\frac{\Delta}{|a|-1}, \frac{\Delta}{|a|-1}\right]$.

First of all, there are two kinds of orbits that stay inside $\overline{\Gamma}$ and leave $\overline{\Gamma}$, respectively. This can be seen from the following example.

Example 1. Consider the system

$$x^+ = 3x - \Delta \mathrm{sgn} x.$$

In this case,

$$\overline{\Gamma} = [-\Delta/2, \Delta/2].$$

The orbit starting from $\frac{5}{12}\Delta$ is

$$\left\{\frac{5}{12}\Delta, \frac{1}{4}\Delta, -\frac{1}{4}\Delta, \frac{1}{4}\Delta, -\frac{1}{4}\Delta, \ldots,\right\},$$

which stays inside $[-\Delta/2, \Delta/2]$.

The orbit starting from $\frac{3}{7}\Delta$ is

$$\left\{\frac{3}{7}\Delta, -\frac{1}{7}\Delta, \frac{4}{7}\Delta, \ldots,\right\},$$

which leaves $\overline{\Gamma}$ (and stays outside $\overline{\Gamma}$), since

$$\frac{4}{7}\Delta \notin [-\Delta/2, \Delta/2].$$

Of course, when an orbit leaves $\overline{\Gamma}$, it leaves there forever. The Lyapunov analysis given in the last subsection can be used to explain why some orbits leave $\overline{\Gamma}$. Remember that along an orbit starting from Ω_c, the Lyapunov function increases from $V(x)$ to the next value $V(x^+)$. We have calculated that the maximal value of $V(x^+)$ on Ω_c is Δ^2. This maximal value is strictly greater than $\frac{\Delta^2}{(|a|-1)^2}$ when $|a|>2$, which is the minimal value of the Lyapunov function outside Γ. That is, there are points in Ω_c such that the immediate next iteration already leaves Ω. In the meantime, we also know that along the orbits starting from $\Gamma\backslash\Omega_c$, the Lyapunov function $V(x)$ decreases. Therefore, there are points in $\Gamma\backslash\Omega_c$, such that after two iterations, the orbits leave Γ, and these points are called unstable points.

The above description can be precisely constructed. Denote the set of unstable points in $\overline{\Gamma}$ as $\mathscr{U}$.

Construction of a cantor set

Iteration 1:

$$\mathscr{U}_1 = \left(-\frac{|a|-2}{|a|(|a|-1)}\Delta, \frac{|a|-2}{|a|(|a|-1)}\Delta\right) \subset \mathscr{U}.$$

This is because, $|x^+| > \frac{\Delta}{|a|-1}$, for any $x \in \mathscr{U}_1$.
Iteration 2:

$$\mathscr{U}_{2,1} \cup \mathscr{U}_{2,2} = \left(-\frac{a^2-2}{a^2(|a|-1)}\Delta, -\frac{a^2-2|a|+2}{a^2(|a|-1)}\Delta\right) \cup \left(\frac{a^2-2|a|+2}{a^2(|a|-1)}\Delta, \frac{a^2-2}{a^2(|a|-1)}\Delta\right) \subset \mathscr{U}.$$

This is because, $x^+ \in \mathscr{U}_1$, for any $x \in \mathscr{U}_{2,1} \cap \mathscr{U}_{2,2}$.
Iteration 3:

$$\bigcup_{k=1}^{4} \mathscr{U}_{3,k} = \left(\pm\frac{|a|\pm 1}{a^2}\Delta - \frac{|a|-2}{|a|^3(|a|-1)}\Delta, \pm\frac{|a|\pm 1}{a^2}\Delta + \frac{|a|-2}{|a|^3(|a|-1)}\Delta\right) \subset \mathscr{U}.$$

...

It is noted that the total length of the intervals in the kth iteration is

$$\frac{2^k(|a|-2)}{|a|^k(|a|-1)}.$$

Therefore, the measure of $\mathscr{U} \subset \overline{\Gamma}$ is

$$\frac{|a|-2}{|a|-1}\sum_{k=1}^{\infty}\left(\frac{2}{|a|}\right)^k = \frac{|a|-2}{|a|-1}\cdot\frac{2}{|a|}\sum_{k=0}^{\infty}\left(\frac{2}{|a|}\right)^k = \frac{2}{|a|-1},$$

which is the measure of $\overline{\Gamma}$.

Let $\mathscr{C} = \overline{\Gamma} \setminus \mathscr{U}$. Then $\mathscr{C}$ is an invariant set of the closed-loop system and it is a Cantor set. To formally prove these and to investigate the behavior of the closed-loop system on $\mathscr{C}$, we make use of some results from fractal geometry [7].

The classic Cantor set is given by taking the interval $[0, 1]$, removing the middle third, removing the middle third of each of the two remaining pieces, and continuing this procedure until infinitum. The Cantor set $\mathscr{C}$ that we are constructing is given by taking the interval $[0, 1]$, splitting it into three intervals, keeping the two with length $1/a$ ($a > 2$), and removing the middle one with length $1 - 2/a$, and continuing forever.

The total length of the line segments in $[0, 1]$ after the kth iteration is $(2/a)^k$, and the number of line segments is $N = 2^k$, so the length of each element is $\epsilon = 1/a^k$. Therefore, *the box dimension* (as well as the Hausdorff dimension) of $\mathscr{C}$ is

$$d = \lim_{k=\infty}\frac{\ln N}{\ln \epsilon} = \frac{\ln 2}{\ln a}.$$

To proceed, we need Theorem 9.1 of [7], which is rephrased as follows.

Theorem 4 [7].

(1) *Let $S_1, \ldots, S_m$ be contracting mappings on $D \subset \mathbb{R}^n$. Then there exists a unique non-empty compact subset F of D such that*

$$F = \bigcup_{i=1}^{m} S_i(F).$$

(2) *For any non-empty compact subset E of D, define*

$$S(E) = \bigcup_{i=1}^{m} S_i(E),$$

and denote by S^k the kth iteration of S. For any non-empty compact set E satisfying $S_i(E) \subset E$, and for each i, it holds that

$$F = \bigcap_{k=1}^{\infty} S^k(E).$$

We adopt the following definition of chaos.

Definition 1 [7]. Let F be an attractor or repeller of the dynamical system (2). The motion of (2) is called chaotic on F if

(i) there is an $x \in F$, such that the orbit $\{f^k(x)\}$ is dense in F;
(ii) the set of periodic points of f in F is dense in F;
(iii) f is sensitive to initial conditions, that is, for any $x, y \in F$, $x \neq y$, there exist a number $\delta > 0$ and an integer k such that

$$|f^k(x) - f^k(y)| \geqslant \delta.$$

We now show that when $|a| > 2$, the Cantor set $\mathscr{C}$ constructed above is a repeller for the closed-loop system f_c, and the motion of f_c on $\mathscr{C}$ is chaotic.

Theorem 5. *If $|a| > 2$, then*

(1) *there is a Cantor set $\mathscr{C}$ with box dimension of $\ln 2/\ln |a|$ in $\overline{\Gamma}$ such that $\mathscr{C}$ is a repeller for the closed-loop dynamical system $f_c(x) = ax - \Delta\,\mathrm{sgn}(ax)$;*
(2) *f_c is chaotic on $\mathscr{C}$.*

Proof. The proof is similar to the development in Section 13.1 of [7] for the tent map. For simplicity, we only prove the case when $a > 2$, and will carry out the proof in two steps.

(1) In order to use Theorem 4, we perform a state transformation,

$$y = \frac{(a-1)x + \Delta}{2\Delta}.$$

The closed-loop system writes in the new state as

$$y^+ = ay - \frac{a-1}{2}(1 + \mathrm{sgn}(2y-1)) \overset{\mathrm{def}}{=} \bar{f}_c(y), \tag{12}$$

and $\overline{\Gamma}$ is transformed into $[0, 1]$ in the new coordinates.
Define

$$S_1(y) = \frac{1}{a}y,$$
$$S_2(y) = \frac{1}{a}y + \frac{a-1}{a}.$$

We see that for $y \in [0,1]$,

$$\bar{f}_c(S_1(y)) = \bar{f}_c(S_2(y)) = y. \tag{13}$$

Note that both S_1 and S_2 are contracting mappings on $[0,1]$. By Theorem 4, there is a unique compact set $\mathscr{C}$ such that

$$\mathscr{C} = S_1(\mathscr{C}) \cup S_2(\mathscr{C}), \tag{14}$$

and $\mathscr{C} = \cap_{k=1}^{\infty} S^k([0,1])$, in which $S([0,1]) = S_1([0,1]) \cup S_2([0,1])$. It is easily seen that $\mathscr{C}$ is a Cantor set, and it is the original Cantor set that we constructed earlier in the subsection, which is now transformed into the new coordinates. The box dimension of this set $\mathscr{C}$ is $\ln 2/\ln a$.
From (13) and (14), $\bar{f}_c(\mathscr{C}) = \mathscr{C}$. So $\mathscr{C}$ is invariant with respect to $\bar{f}_c$.
To show that $\mathscr{C}$ is a repeller of $\bar{f}_c$, note that if $y < 0$, then

$$y^+ = \bar{f}_c(y) = ay,$$

so when $k \to \infty$, $\bar{f}_c^k(y) \to -\infty$. If $y > 1$, then

$$y^+ = \bar{f}_c(y) = ay - a + 1,$$

so when $k \to \infty$, $\bar{f}_c^k(y) \to \infty$. If $y \in [0,1] \setminus \mathscr{C}$, then there exists an integer k such that $y \notin \cup\{S_{i_1} \circ \cdots \circ S_{i_k}[0,1]: \ i_j = 1,2\}$, so $\bar{f}_c^k(y) \notin [0,1]$. Therefore, either $\bar{f}_c^k(y) \to -\infty$ or $\bar{f}_c^k(y) \to \infty$ when $k \to \infty$. That is, $\mathscr{C}$ is a repeller of $\bar{f}_c$.

(2) Since

$$\mathscr{C} = \bigcap_{k=1}^{\infty} S^k([0,1]) = \bigcap_{k=1}^{\infty} \bigcup_{(i_k=1,2)} \{S_{i_1} \circ \cdots \circ S_{i_k}[0,1]\},$$

we can use $y_{i_1,i_2,\ldots}$, $i_j = 1,2$, to represent those points in $\mathscr{C}$ that were obtained after iterations $i_1, i_2, \ldots$ If $i_1 = i'_1, \ldots, i_k = i'_k$, then

$$|y_{i_1,i_2,\ldots} - y_{i'_1,i'_2,\ldots}| \leqslant 1/a^k. \tag{15}$$

Note that $\bar{f}_c(y_{i_1,i_2,\ldots}) = y_{i_2,i_3,\ldots}$, since $y_{i_1,i_2,\ldots} = S_{i_1}(y_{i_2,i_3,\ldots})$. For any point $y_{i'_1,i'_2,\ldots} \in \mathscr{C}$ and any integer q, we can find k such that $(i'_1, i'_2, \ldots, i'_q) = (i_{k+1}, \ldots, i_{k+q})$. So $\bar{f}_c^k(y_{i_1,i_2,\ldots}) = y_{i_{k+1},i_{k+2},\ldots}$ By (15),

$$|\bar{f}_c^k(y_{i_1,i_2,\ldots}) - y_{i'_1,i'_2,\ldots}| \leqslant 1/a^k.$$

Hence, $\bar{f}_c$ has a dense orbit in $\mathscr{C}$.On the other hand, $y_{i_1,\ldots,i_k,i_1,\ldots,i_k,i_1,\ldots}$ is a periodic point with period k, so the set of periodic points of $\bar{f}_c$ is dense in $\mathscr{C}$.Because $\bar{f}_c^k(y_{i_1,\ldots,i_k,1,\ldots}) \in [0,1/a]$ but $\bar{f}_c^k(y_{i_1,\ldots,i_k,2,\ldots}) \in [1-1/a,1]$, it shows that $\bar{f}_c$ is sensitive to initial conditions.This concludes that the motion of $\bar{f}_c$ is chaotic on $\mathscr{C}$. □

3. Conclusions

In this paper, we have investigated some interesting and quite complex properties of a rather simple system controlled by Δ-modulated feedback. We have shown that there are three different cases. In the first case, there is a minimal global attractor consisting of only two points. The two points form either one 2-periodic orbit or two 1-periodic orbits (fixed points). We have also characterized the attracting region for each of these two points. In the second case, the stabilizable region of the system is bounded, and there is a local attractor inside this stabilizable region. In the third case, the maximal stabilizable set of the system is a Cantor set, and this Cantor set is a repeller of the system. Moreover, the system is chaotic on the Cantor set.

We remark that the currently available theory of continuous dynamic systems does not apply to our case where the Δ-modulated feedback introduces discontinuity. For example, we have found an example with 2-periodic points but without 1-periodic points. This is a major departure from what the famous Sarkovskii theorem claims for continuous dynamic systems. Due to the discontinuity, the periodic points of the Δ-modulated feedback system in the case of $1 < |a| \leqslant 2$, and the behavior of the closed-loop system in the attractor $[-\Delta, \Delta]$, could be more complicated than the continuous case [14].

In the development of these results, we have applied various analytical methods of different mathematical sophistication. The intention of doing so is to shed some lights for possible generalization of the results to higher-dimensional systems. Indeed, many results can be generalized by using the Lyapunov approach, which will be reported elsewhere.

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References

[1] Ariyavisitakul S, Chang L. Simulation of a CDMA system performance with feed-back power control. Electron Lett 1991;27:2127–8.
[2] Boyarsky A, Gora P. Laws of chaos: invariant measures and dynamical systems in one dimension (probability and its applications). Berlin: Springer-Verlag; 1997.
[3] Brockett RW, Leberzon D. Quantized feedback stabilization of linear systems. IEEE Trans Automat Contr 2000;45:1279–89.
[4] Candy JC. A use of limit cycle oscillations to obtain robust analog-to-digital converters. IEEE Trans Commun 1974;22:298–305.
[5] Delchamps DF. Stablilizing a linear system with quantized state feedback. IEEE Trans Automat Contr 1990;35:916–24.
[6] Elia N, Mitter SK. Stabilization of linear systems with limited information. IEEE Trans Automat Contr 2001;46:1384–400.
[7] Falconer K. Fractal geometry: mathematical foundations and applications. London: John Wilay; 1989.
[8] Feely O, Chua LO. Nonlinear dynamics of a class of analog-to-digital converters. Int J Bifurcat Chaos 1992;2:325–40.
[9] Inose H, Yasuda Y. A unity bit coding method by negative feedback. Proc IEEE 1963;51:1524–35.
[10] Leonov NN. Map of the line into itself. Radiofisica 1959;2:942–56.
[11] Liberti Jr JC, Rappaport TS. Smart antennas for wireless communications: IS-95 & third generation CDMA applications. New Jersey: Prentice Hall; 1999.
[12] Park SJ, Gray RM. Sigma–delta modulation with leaky integration and constant input. IEEE Trans Inform Theory 1992;38:1512–33.
[13] Sarkovskii AN. Coexistence of cycle of a continuous map of a line into itself. Ukr Mat Z 1964;16:61–71.
[14] Xia X, Gai R, Chen G. Periodic orbits arising from Delta-modulated feedback control. Chaos, Solitons & Fractals 2004;19:581–95.
[15] Xia X, Zinober ASI. Δ-modulated feedback in discretization of sliding mode control. Automatica, in press.